NOTES AND QUERIES.

MR. CHARLES GABRIEL SHAW, an Associate of the Institute, sends the following ingenious method of obtaining formulæ for the present value and for the amount of an annuity for n years, commencing at a and increasing by c yearly, analogous to those given by Mr. E. H. Galsworthy in the last Number of the *Magazine.*

Let r equal the yearly interest of £1, and let $v=\frac{1}{1+r}$, and $A_n=\frac{1-v^n}{r}$.

The present value of the annuity will be, $av+(a+c)v^2+(a+2c)v^3+$, &c. $+(a+\overline{n-1}c)v^n$.

When $n=\infty$, or is infinite, and the annuity becomes a perpetuity, this will be $\frac{a}{r}+c\{v^2+2v^3+3v^4+$, &c., *ad infinitum*$\}$.

Calling the series within the brackets S, then $(1+r)S=\{v+2v^2+3v^3+4v^4+$, &c.$\}$, and $rS=\{v+v^2+v^3+v^4+$, &c.$\}=\frac{1}{r}$, and $S=\frac{1}{r^2}$; then the perpetuity equals $\frac{a}{r}+\frac{c}{r^2}$.

When n is finite, the value of all the terms of the perpetuity after the nth must be subtracted from the above. At the beginning of the $\overline{n+1}$st year, the next payment due being $(a+nc)$, and the future increase still c per annum, the value of the perpetuity will be, by the above formula, $\frac{a+nc}{r}+\frac{c}{r^2}$, which must be multiplied by v^n to bring it to the *present* value of all that part of the perpetuity which is payable after the expiry of n years. Effecting this subtraction, we find the value as required of the first n payments,

$$\frac{a}{r}+\frac{c}{r^2}-\left(\frac{a+nc}{r}+\frac{c}{r^2}\right)v^n=\left(\frac{1-v^n}{r}\right)a+\left(\frac{1-v^n}{r^2}\right)c-\frac{nv^nc}{r}$$
$$=A_na+\left(\frac{A_n-nv^n}{r}\right)c.$$

As to the amount of the annuity, it must of course be the same as that of the present value accumulated for n years; and if we multiply that value by $(1+r)^n$, and make $M_n=\frac{(1+r)^n-1}{r}$, we have for this amount $M_na+\left(\frac{M_n-n}{r}\right)c$.

THE following is the problem referred to at page 134, Vol. IV., as sent by Mr. JAMES MEIKLE, of the Scottish Provident Institution:—

To determine the rate of interest in a life annuity, the table of mortality and age being given.

$A = {}_1av^1 + {}_2av^2 + {}_3av^3 + \dots + {}_nav^n.$

It is evident that any series which represents the value of v will converge very slowly; but by making $v = 1 - d$, we get

$$\begin{aligned} A. &= {}_1a(1-d)^1 + {}_2a(1-d)^2 + {}_3a(1-d)^3 + \dots + {}_na(1-d)^n \\ &= {}_1a + {}_2a + {}_3a + \dots + {}_na \\ &\quad - d\{{}_1a + 2{}_2a + 3{}_3a + \dots + n{}_na\} \\ &\quad + d^2\left\{{}_2a + 3{}_3a + 6{}_4a + \dots + \frac{n.n-1}{1.2}{}_na\right\} \\ &\quad - d^3\left\{{}_3a + 4{}_4a + 10{}_5a + \dots \frac{n.n-1.n-2}{1.2.3}{}_na\right\} \\ &\quad +, \text{ \&c.} \end{aligned}$$

But

$$\left.\begin{aligned} {}_1a + {}_2a + {}_3a + \dots + {}_4a &= \frac{N_x}{D_x} \\ {}_1a + 2{}_2a + 3{}_3a + \dots + n{}_na &= \frac{S_x}{D_x} \\ {}_2a + 3{}_3a + 6{}_4a + \dots \frac{n.n-1.}{1.2}{}_na &= \frac{\Sigma S_{x+1}}{D_x} \\ \text{\&c.} \qquad \text{\&c.} & \end{aligned}\right\} \begin{aligned} &\text{where } D_x = \text{No. living at age } x \\ &N_x = \text{Sum of living at } x \text{ and upwards} \\ &S_x = \text{Sum of } A_x \\ &\Sigma S_{x+1} = \text{Sum of } S_{x+1}, \text{ \&c.} \end{aligned}$$

$$\therefore\ A = \frac{A_x}{D_x} - \frac{dS_x}{D_x} + \frac{d^2\Sigma S_{x+1}}{D_x} - \frac{d^3\Sigma^2 S_{x+2}}{D_x} +, \text{ \&c.}$$

Now the coefficients of d are constant for any value of d, and independent of it.

Let $\frac{N_x}{D_x} - A = a$; $\frac{S_x}{D_x} = b$; $\frac{\Sigma S_{x+1}}{D_x} = c$; $\frac{\Sigma^2 S_{x+2}}{D_x} = d$, &c. Then the series may assume the form of $a - bd + cd^2 - dd^3 + ed^4 - fd^5$, &c. $= o$; or, putting $d = y$, for the sake of distinction, $a - by + cy^2 - dy^3 + ey^4 - fy^5 +$, &c. $= o$; from which the value of y may be found by Lagrange's theorem.

$$\left.\begin{aligned} u &= f(y) \\ y &= z + x\phi(y) \end{aligned}\right\} \therefore u = f(z) + \frac{x}{1.2}\phi(z)\frac{df(z)}{dz} + \frac{x^2}{1.2}\frac{d}{dz}\left\{\overline{\phi(z)}|^2\frac{df(z)}{dz}\right\}$$

$$+ \frac{x^3}{1.2.3}\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3\frac{dfz}{dz}\right\} +, \text{ \&c.}$$

$$f(z) = z = \frac{a}{b}$$

$$\phi(z) = \frac{cz^2 - dz^3 + ez^4 -, \text{ \&c.}}{b}, \text{ \&c.}$$

$$y = \frac{a}{b} + \frac{cz^2}{b} + \frac{2c^2 - bd}{b^2}z^3 + \frac{5c^3 - 5cdb + eb^2}{b^3}z^4 + \frac{bceb^2 - fb^3 - 21c^2db + 14c^4 + 3d^2b^2}{b^4}z^5$$
$$+ \frac{gb^4 - 7cfb^3 - 7deb^3 + 28c^2eb^2 + 28cd^2b^2 - 84c^3db + 42c^5}{b^5}z^6 +, \text{ \&c.}$$

$\therefore$

$$d = \frac{a}{b} + \frac{ca^2}{b^3} + \frac{2c - bd}{b^5}a^3 + \frac{5c^3 - 5cdb + eb^2}{b^7}a^4 + \frac{6ceb^2 - fb^3 - 21c^2db + 14c^4 + 3d^2b^2}{b^9}a^5$$
$$+ \frac{gb^4 - 7cfb^3 - 7deb^3 + 28c^2eb^2 + 28cd^2b^2 - 84c^3db + 42c^5}{b^{11}}a^6 +, \text{ \&c.}$$

This series converges rapidly, as will be seen from the following example:—

Find the rate of interest in the sum of £4·18289, which is the value of a life annuity on age 80, by Carlisle Table of Mortality.

$$
\begin{aligned}
\frac{a}{b} &= \cdot 0340012 \\
\frac{ca^2}{b^3} &= \cdot 00388575 \\
\&c. &= \cdot 00049785 \\
\&c. &= \cdot 000065931 \\
\&c. &= \cdot 0000088845 \\
\&c. &= \underline{\cdot 0000012315} \\
d &= \cdot 0384608470 \\
\text{nearly} &= \underline{\cdot 0384615}
\end{aligned}
\qquad 1-d=v=\cdot 961548=\frac{1}{1+r}
$$

$$\therefore\ r=\frac{1-v}{v}=\frac{d}{v}=\frac{\cdot 0384615}{\cdot 9615385}=\cdot 04$$

∴ Rate of interest=4 per cent.

MR. MEIKLE also furnishes the following:—

Assurances are effected either by premiums payable for the whole of life or for a term of years. A person opens a policy on the whole life scale, and, after a few years, proposes to change to the other scale. What sum ought he to pay, to compensate for the previous deficiencies? and of what is it composed?

Let the case be illustrated by an example:—Entrant of 30, paying 17·5541 (Carlisle 4 per cent. pure), at age 35 proposes to change to the 21 years' scale, the premium on which would have been 23·6069.

$\frac{17\cdot 5541(1+a_{35})}{{}_{\overline{16}|}a_{35}+1-{}_{16}av^{16}}$ =Future life premiums of 17·5541 converted into a term of 16 years . . £26·7535

But he only intends paying . . . . 23·6069

and therefore must redeem this . . . . 3·1466

which will be the sum required, $3\cdot 1466({}_{\overline{16}|}a_{35}+1-{}_{16}av^{16})=35\cdot 184$.

Or, analytically—Let $[\cdot]$=life premium; $[21]$=21 years' premium. Then sum required=$S=[\cdot](1+a_{35})-[21]({}_{\overline{16}|}a_{35}+1-{}_{16}a_{35}v^{16})$.

$$S\,{}_5a_{30}v^5=[\cdot](1+a_{35})\,{}_5a_{30}v^5-[21]({}_{\overline{16}|}a_{35}+1-{}_{16}a_{35}v^{16})\,{}_5a_{30}v^5$$

$$=[\cdot](1+a_{30})-[\cdot]({}_{\overline{5}|}a_{30}+1-{}_5a_{30}v^5)-[21]\,{}_{\overline{21}|}a_{30}+1-{}_{21}a_{30}v^{21}+[21]({}_{\overline{5}|}a_{30}+1-{}_5a_{30}v^5)$$

$$S=\frac{([21]-[\cdot])\,{}_{\overline{5}|}a_{30}+1-{}_5a_{30}v^5}{{}_5a_{30}v^5}$$

$$=([21]-[\cdot])\left\{\frac{{}_0a_{30}v^0+{}_1a_{30}v^1+{}_2a_{30}v^2+{}_3a_{30}v^3+{}_4a_{30}v^4}{{}_5a_{30}v^5}\right\}$$

$$=([21]-[\cdot])\left\{\frac{1}{{}_5a_{30}v^5}+\frac{1}{{}_4a_{31}v^4}+\frac{1}{{}_3a_{32}v^3}+\frac{1}{{}_2a_{33}v^2}+\frac{1}{{}_1a_{34}v^1}\right\}$$

Proving that each yearly deficiency is increased by the chance of death during nonpayment, besides the ordinary interest.

Demonstration of the expressions $\frac{(1+i)^n-1}{i}$ *the Amount, and* $\frac{1-(1+i)^{-n}}{i}$ *the Present Value, of £1 per Annum for n Years.*—Since $(1+i)^n-1$ evidently represents the amount of an annuity of i pounds in n years, the first formula above given is clearly that for the *amount* of £1 per annum for the same term. For as $i : (1+i)^n-1 :: 1 : \frac{(1+i)^n-1}{i}$; in like manner, $\frac{1-(1+i)^{-n}}{i}$ is the *present value* of £1 per annum for n years. For as $(1+i)^n : 1 :: \frac{(1+i)^n-1}{i} : \frac{(1+i)^n-1}{i.(1+i)^n} = \frac{1-(1+i)^{-n}}{i}$.
—Ed. *A. M.*

FOREIGN INTELLIGENCE.

Germany.—*New Insurance Companies in Germany.*—The General Railway Insurance Company in Berlin (*Allgemeine Eisenbahn Versicherungs Gesellschaft*), registered by Order of Council of His Majesty the King of Prussia, of the 26th September, 1853, with a subscribed capital of Prussian thalers 1,000,000 (£166,000 sterling), in 1,000 shares, of which 20 per cent. is paid up, insures, according to the deed of constitution—

Class 1. To railway managers, for any loss and damage on moveable and immovable property, or on articles and goods of all kinds to be forwarded by trains or to be kept at the stations.

Class 2. To passengers and railway officers, for any personal injury or loss of life, and for loss on luggage.

The prospectus issued treats only of this second class, and I extract the following particulars.

Insurance on persons is granted to railway passengers, either for single days, for one, three, and six months, for one or five years, or for the duration of life, along all the railways of Europe; to railway officers, only for the period of one year, at the premium of 1 per cent.

The maximum to be insured by a passenger is Pr. th. 7,000 (£1,150); by an officer, Pr. th. 1,000 (£160).

Claims are settled according to the following rates—

100 per cent.	in case the	accident	causes	death immediately, or within 30 days.
Up to $\frac{1}{3}$ „	do.	do.		incapacity to work for 8 days.
„ $6\frac{2}{3}$ „	do.	do.		„ for a longer time.
„ $33\frac{1}{3}$ „	do.	do.		loss of limb, or lasting serious damage to health.
„ $66\frac{2}{3}$ „	do.	do.		total incapacity to work.